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The Complex Potential Generated by the Maximal Measure for a Family of Rational Maps

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The exact complex potential generated by the maximal measure for a family of rational maps is given. The results are of analytical nature because the complex potential does not change nicely if the coordinates of a rational map are changed. There exist applications of this result to the theory of moments.

KEY WORDS: Maximal measure; balanced measure; rational maps; complex potential; theory of moments; Julia sets.

1. INTRODUCTION

Consider $f(z)$ a rational map on the Riemann sphere of degree d of the form

$$f(z)=\frac{p(z)}{q(z)}=\frac{z^d+a_{d-1}z^{d-1}+\cdots+a_1z+a_0}{b_kz^k+\cdots+b_1z+b_0}$$

where $a_i\in\mathbb{C}$, $b_j\in\mathbb{C}$, $i\in\{0,..,d-1\}$, $j\in\{1,...,k\}$, and $d>k$. Suppose also the Julia set of f (see ref. 5) is bounded in the complex plane.

Let z_0 be a point on the complex plane and for each $n\in N$, $i\in\{1,..., d^n\}$, let z_i^n be the d^n solutions of $f^n(z)=z_0$.

Consider now

$$u_n=\frac{1}{d^n}\sum_{i=1}^{d^n}\delta_{z_i^n}$$

where δ_x is the Dirac measure on x.

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In refs. 7 and 10 it is shown that there exists $u = \lim_{n \to \infty} u_n$ and this measure u is the measure of maximal entropy. Some authors call this measure u of balanced measure.[1-3]

As is very well known, the Julia set of a rational map can have a very complicated structure and has in general a fractal dimension.[5,11] The maximal measure of a rational map, therefore, can have support in a very strange subset of the plane. It would be very difficult to give an explicit analytic expression for such a measure.

A nicer object to work with is the potential function $F(z)$,[1,8] which satisfies: $F(z)$ is analytic around ∞, $F(\infty) = \infty$, $F'(\infty) = 1$, and for $z \sim \infty$

$$\log |F(z)| = \int \log |z - x| \, du(x)$$

In ref. 8 it is shown that for $z \sim \infty$, $F(z)$ satisfies the functional equation

$$F(f(z)) = \frac{F(z)^d}{q(z)}$$

when

$$f(z) = \frac{p(z)}{q(z)} = \frac{z^d + \cdots + a_0}{b_k z^k + \cdots + b_0}$$

Here we will show that for a certain family of rational maps, given $f(z)$, we can solve the functional equation and obtain $F(z)$.

Theorem. Let $f(z)$ be a rational map of the form

$$f(z) = \frac{z^2 + cz + d}{mz + n} = \frac{p(z)}{q(z)}$$

where $|m| < 1$ (in order to have the Julia set bounded). In the case there exist $e, f \in \mathbb{C}$ such that

$$\left.\begin{aligned}
&\text{(I)} \quad e^2 = en^2 + fdn + d^2 \\
&\text{(II)} \quad 2ef = 2emn + (dm + cn)f + 2dc \\
&\text{(III)} \quad (2e + f^2) = em^2 + f(cm + n) + (2d + c^2) \\
&\text{(IV)} \quad 2f = fm + 2c
\end{aligned}\right\} \qquad (*)$$

then $F(z) = (z^2 + fz + e)^{1/2}$ is a solution of the functional equation

$$F(f(z)) = \frac{F(z)^2}{q(z)} = \frac{F(z)^2}{mz + n}$$

Remark. For $f(z)$ a polynomial map, we will not obtain solutions that are not trivial ones [all conjugated to $z^2 = f(z)$].

The proof of the theorem is given in Section 3. Some applications of the theorem are given in Section 2.

Brolin[5] was the first to notice the electrostatic properties of the Julia set, by showing that for polynomials, the balanced measure as defined in the introduction is the charge distribution in the Julia set.[13]

In ref. 8 it is shown that for rational maps such that $f(\infty) = \infty$ and the Julia set is bounded, then the maximal measure is always different from the charge distribution. In ref. 6 it is shown that the charge distribution in the Julia set can be obtained by a generalization of the procedure of Brolin.

As a final remark, I mention the fact that the complex potential does not change by means of a change of coordinates if one performs a change of coordinates for the rational map.

2. THE MOMENT PROBLEM

For a given measure u, the moments of the measure u are by definition

$$M_k = \int z^k \, du(z) \qquad \text{for} \quad k \in \mathbb{N}$$

The problem of computing the moments of a certain measure is a classical subject in the theory of orthogonal polynomials, Padé approximants, and potential theory.[12,13]

Some authors have already considered the problem of obtaining the moments of the maximal measure.[1–4,8] In general this problem is associated with a three-term relation.

If one knows the potential function $F(z)$, one can obtain the moments in the following way: for $z \sim 0$ we have

$$\begin{aligned} z^{-1} \frac{F'(z^{-1})}{F(z^{-1})} &= z^{-1} \int (z^{-1} - x)^{-1} \, du(x) \\ &= \int \frac{1}{1 - xz} \, du(x) = \sum_{n=0}^{\infty} \left[\int x^n \, du(x) \right] z^n = \sum_{n=0}^{\infty} M_n z^n \end{aligned}$$

The theorem announced in Section 1 allows one to obtain the explicit value of the function $F(z)$ and therefore the moments, as observed above.

Assuming $d = 0$ in $(*)$, we obtain $f(z)$ of the form

$$f(z) = \frac{z^2 + (2 - m)nz}{mz + n}, \qquad 0 < |m| < 1$$

and we obtain $F(z) = z + u$.

Therefore $F'(z)/F(z) = 1/(z+n)$ and the moments are $M_k = n^k$.

Note that one can obtain this information just by knowing the coefficients of the rational map, without knowing exactly this measure. This shows the utility of the function $F(z)$.

The information given by the moments is very useful if one wants to obtain approximations of the values of the integral of a certain function $\varphi(z)$ with respect to u. One just has to consider good approximations of $\varphi(z)$ by polynomials and use the moments.

For the maximal measure, for instance, this method is more suitable for computations that try to use directly the definition of the measure given in Section 1. For a rational map of degree larger than four this method of moments is particularly more suitable.

3. PROOF OF THE THEOREM

Proof. Suppose $f(z) = (z^2 + cz + d)/(mz + n)$ and $F(z) = (z^2 + fz + e)^{1/2}$, and make the change of coordinates z^{-1}. Denote by $f(z)$ and $F(z)$ the corresponding functions in the new variables for $z \sim 0$. Therefore

$$f(z) = \frac{nz^2 + mz}{dz^2 + cz + 1}, \qquad F(z) = \frac{z}{(ez^2 + fz + 1)^{1/2}}$$

and the new functional equation is

$$F(f(z)) = F(z)^2 \frac{m + nz}{z}$$

The first term is therefore

$$\begin{aligned} &F(f(z)) \\ &\quad = (nz^2 + mz)/(dz^2 + cz + 1) \\ &\qquad \times \left[\frac{e(nz^2 + mz)^2 + f(nz^2 + mz)(dz^2 + cz + 1) + (dz^2 + cz + 1)^2}{(dz^2 + cz + 1)^2}\right]^{-1/2} \\ &\quad = \frac{nz^2 + mz}{[e(nz^2 + mz)^2 + f(nz^2 + mz)(dz^2 + cz + 1) + (dz^2 + cz + 1)^2]^{1/2}} \end{aligned}$$

The second term is

$$F(z)^2 \frac{m + nz}{z} = \frac{z^2}{ez^2 + fz + 1} \frac{m + nz}{z} = \frac{nz^2 + mz}{ez^2 + fz + 1}$$

The result follows if

$$(ez^2 + fz + 1)^2 = e(nz^2 + mz)^2 + f(nz^2 + mz)(dz^2 + cz + 1) + (dz^2 + cz + 1)^2$$

The four relations (I)–(IV) are the equations that have to be satisfied by the coefficients of z, z^2, z^3, z^4 for the two four-degree polynomials above.

From ref. 9, $F(z)$ is the unique solution of the equation such that $F'(\infty) = 1$.

This is the end of the proof of the theorem.

Remark 1. One can obtain e and f from c, d, m, and n linearly from (II) and (IV).

Remark 2. After this paper was written, P. Moussa communicated to me that if one considers $g(z) = z^2 + cz$, $c \in \mathbb{C}$, and makes a change of coordinates z^{-1} followed by a translation, one obtains $f(z)$ in the example considered here when $d = 0$ (see Section 2). In this case he already knew the solution $F(z) = z + n$.

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